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Stress Singularity at a Sharp Edge in Contact Problems with Friction

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Introduction

Geometric singularities constitute a fairly well explored area in linear elasticity. The earliest studies done by Williams [1, 2] start with an assumption about the general structure of the singularity, and then establish by calculation when such a singularity exists and explore its more precise nature. The hazard in this approach is that possible singular states can be missed. A safer and more systematic method is to use a Mellin transform, which was first employed for this purpose in linear elasticity by Sternberg and Koiter [3]. A very thorough study of geometric singularities in bodies consisting of two different materials has since been carried out by Bogy and Wang [4, 5].

Geometric singularities arise also in contact problems when one of the bodies has a sharp edge. The frictionless contact has been treated by Dundurs and Lee [6]. The effect of friction has more recently been investigated by Gdoutos and Theocaris [7]. Both studies take the body with the smooth surface as an elastic half space and the body with the sharp corner as a wedge. It was shown by Dundurs and Lee using a Mellin transform that no singularities develop for wedges that are relatively soft in comparison to the half space and have small angles. In contrast, power singularities arise for more blunt wedges made of a stiffer material. Furthermore, a logarithmic singularity separates the regions of no singularities and power singularities. The investigation by Gdoutos and Theocaris, which employs the Williams technique, concludes that no singularities may appear in contact with friction under circumstances similar to the frictionless case, provided the bodies slip with respect to each other. The purpose of this note is to explore the nature of the singularity under frictional slip in more detail, and to display experimental results showing that the difference between very low and very high friction clearly shows in photoelastic tests.

Theoretical Results

The use of the Mellin transform in studying the nature of geometric singularities has been explained by Bogy [4], and hence we give only the results. The possible singular states in the vicinity of the vertex of the wedge are determined by the character of the determinant $\mathscr{D}$ which appears in the Mellin transform of the elastic fields. The following criterion for judging the nature of the singularity may be repeated from the aforementioned paper by Bogy: If p is a zero of $\mathscr{D}$ in the strip $0 < \mathrm{Re}(p) \leq 1$, the orders of the singularities in the stresses as $r \to 0$ are:

$$
\begin{aligned}
\sigma_{ij} = {} & 0(r^{p-1}) \quad \text{for } p \text{ real and } \quad 0 < p < 1, \\
& 0[r^{\xi-1}\cos(\eta\log r)] \quad \text{or} \quad [r^{\xi-1}\sin(\eta\log r)] \\
& \qquad\qquad \text{for} \quad p = \xi + i\eta \quad \text{complex and} \quad 0 < \xi < 1, \\
& 0(\log r) \quad \text{for} \quad p = 1 \quad \text{and} \quad \partial\mathscr{D}/\partial p = 0 \quad \text{at} \quad p = 1, \\
& 0(1) \quad \text{for no zeroes of } \mathscr{D} \text{ in } 0 < \mathrm{Re}(p) < 1 \quad \text{and} \quad \partial\mathscr{D}/\partial p \neq 0 \quad \text{at} \quad p = 1. \qquad (1)
\end{aligned}
$$

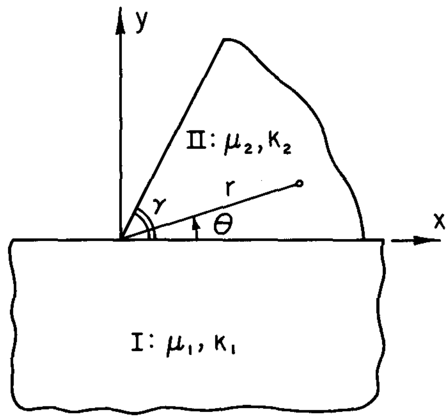

Figure 1
Wedge in contact with a half space.

We use the same notation as in the paper by Dundurs and Lee [6] which treats the frictionless case; the labeling of the materials is shown in Fig. 1. The boundary conditions at the interface between the half space and the wedge specify continuity of normal displacements and normal tractions, and incorporate the Coulomb law of dry friction. *The nature of the singularity at the vertex of the wedge depends on the direction in which the wedge slips with respect to the half space.* However, both possible directions can be treated at the same time by letting the coefficient of friction assume also negative values. Thus, if ρ denotes the coefficient of friction, positive values of ρ correspond to the wedge slipping with respect to the half space in the positive direction of x, while negative values of ρ indicate slip in the opposite direction. This becomes clear from the boundary condition

$$\sigma_{r\theta} = -\rho\sigma_{\theta\theta}, \qquad \sigma_{\theta\theta} < 0 \tag{2}$$

on $x \geq 0$, $y = 0$ by considering the customary sign convention for $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$.

Following the same steps as in the paper by Bogy [4], a rather tedious calculation yields the determinant in the Mellin transform of the elastic field as

$$\mathscr{D}(p; \gamma, \alpha, \beta, \rho) = 8(1 + p) \sin p\pi F(p; \gamma, \alpha, \beta, \rho) \tag{3}$$

where

$$F(p; \gamma, \alpha, \beta, \rho) = (1 + \alpha) \cos p\pi(\sin^2 p\gamma - p^2 \sin^2 \gamma) + \tfrac{1}{2}(1 - \alpha) \sin p\pi(\sin 2p\gamma + p \sin 2\gamma)$$
$$+ \rho \sin p\pi[(1 - \alpha)p(1 + p) \sin^2 \gamma - 2\beta(\sin^2 p\gamma - p^2 \sin^2 \gamma)] \tag{4}$$

and γ is the wedge angle. Furthermore,

$$\alpha = \frac{(\mu_2/\mu_1)(\kappa_1 + 1) - (\kappa_2 + 1)}{(\mu_2/\mu_1)(\kappa_1 + 1) + \kappa_2 + 1}, \qquad \beta = \frac{(\mu_2/\mu_1)(\kappa_1 - 1) - (\kappa_2 - 1)}{(\mu_2/\mu_1)(\kappa_1 + 1) + \kappa_2 + 1} \tag{5}$$

are the parameters introduced by Dundurs [8] characterizing the mismatch in the elastic constants of the two materials, in which $\kappa = 3 - 4\nu$ for plane strain, with ν denoting Poisson's ratio. Taking into account the different labeling of the materials and sign convention on ρ, the function $F(p; \gamma, \alpha, \beta, \rho)$ is seen to be essentially the same as that given by Eq. (8) in the paper by Gdoutos and Theocaris [7].

The appearance of oscillating singularities is connected with complex roots of $F(p; \gamma, \alpha, \beta, \rho)$ in the strip $0 < \mathrm{Re}(p) < 1$. For $\rho = 0$, this function is

$$F(p; \gamma, \alpha, \beta, 0) = (1 + \alpha) \cos p\pi(\sin^2 p\gamma - p^2 \sin^2 \gamma) + \tfrac{1}{2}(1 - \alpha) \sin p\pi(\sin 2p\gamma + p \sin 2\gamma) \tag{6}$$

which, except for the opposite sign, is the same as the determinant given by Eq. (4) in the paper by Dundurs and Lee [6]. Dundurs and Lee remark that it does not seem possible to prove analytically the absence of complex roots in the strip $0 < \mathrm{Re}(p) < 1$. They claim, however, that a systematic numerical study has provided convincing evidence that no such complex roots exist. Since $F(p; \gamma, \alpha, \beta, \rho)$ is much more complicated than $F(p; \gamma, \alpha, \beta, 0)$ the question of complex roots will be left open in this note.

Power singularities are related to the real roots of $F(p; \gamma, \alpha, \beta, \rho)$ in the interval $0 < p < 1$. A numerical study of these roots for $\gamma = 60°$, $90°$ and $180°$ was carried out by Gdoutos and Theocaris [7]. Their display of the results in the Dundurs parallelogram [8] for discrete values of ρ reveals that, similarly to the frictionless case, no power singularities appear for certain combinations of wedge angles and elastic constants. It is possible, however, to express analytically the demarcation between power singularities and no power singularities. Note first that $F(1; \gamma, \alpha, \beta, \rho) = 0$ for all combinations of $\gamma, \alpha, \beta,$ and ρ. Suppose now that $0 < p = p_1 < 1$ is a root of $F(p; \gamma, \alpha, \beta, \rho) = 0$ for a set of values of γ, α, β and ρ. Since F is a continuous and differentiable function of p, it has then a relative extremum between $p = p_1$, and $p = 1$. Next suppose that, say, γ is varied for fixed values of α, β and ρ so that $p_1 \to 1$. In such case the relative extremum also approaches $p = 1$. Furthermore, the study of the roots of $F(p; \gamma, \alpha, \beta, \rho)$ for discrete values of the physical parameters in [7] indicates that there is at most one real root in the interval $0 < p \leq 1$. Hence the locus separating the regimes of power singularities and no power singularities corresponds to

$$\frac{\partial F(1; \gamma, \alpha, \beta, \rho)}{\partial p} = 0. \tag{7}$$

Applying (7) to (4), the result is

$$\alpha = \frac{(\pi + \gamma)\cos\gamma + (\pi\rho - 1)\sin\gamma}{(\pi - \gamma)\cos\gamma + (\pi\rho + 1)\sin\gamma}. \tag{8}$$

The results based on (8) are displayed in Fig. 2 for discrete values of the friction coefficient ρ. To the left and below any of the curves $\rho = \text{const}$ (i.e., when $=$ is replaced with $\leq$ in the last expression), no power singularities are possible, whereas they may appear to the right and above these loci. The rather unexpected result is that, depending on the direction of relative slip between the bodies, friction may prevent the appearance of power singularities and thus alleviate stress concentration effects. It is seen that slip of the wedge in the positive direction of x has a beneficial effect, but that slip in the opposite direction is detrimental. One should realize, however, that the direction of slip depends on factors that do not enter into a study of possible singular states. In fact, the direction of slip is one of the unknowns in the general problem when two finite bodies are pressed together by, say, specified surface tractions. It also is possible that the bodies do not slip at all. In such case the singularities for perfect bond studied by Bogy [4], and Gdoutos and Theocaris [7] will appear at the sharp edge.

Perhaps the most important additional observation that can be made on basis of the determinant in the Mellin transform given by (3) pertains to the logarithmic singularities. Differentiating (3),

$$\frac{\partial \mathscr{D}}{dp} = 8\left\{(1 + p)\sin p\pi \frac{\partial F}{\partial p} + [\pi(1 + p)\cos p\pi + \sin p\pi]F\right\}. \tag{9}$$

From (9) and (3), it is seen that $\partial\mathscr{D}/\partial p = \mathscr{D} = 0$ for $p = 1$ and, consequently *logarithmic*

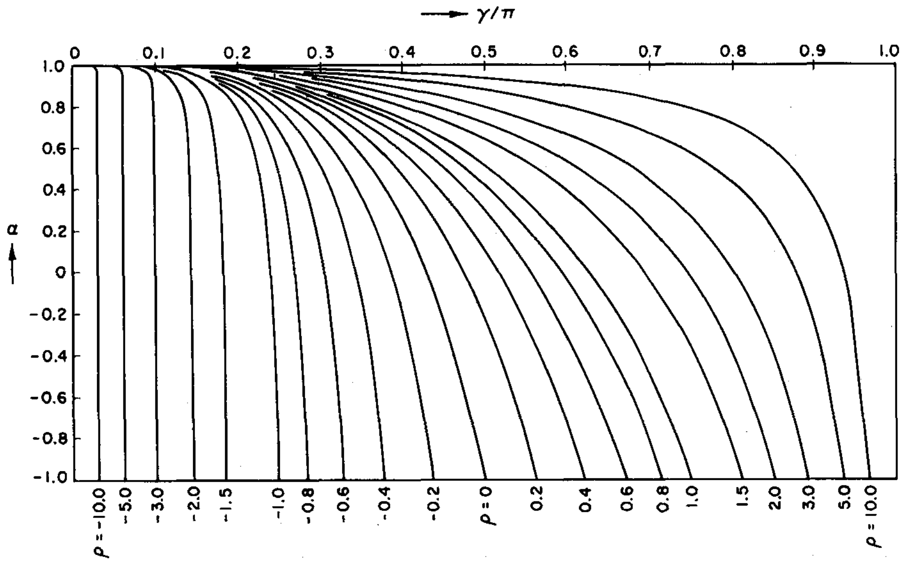

Figure 2
Loci separating regions of power and no power singularities.

singularities are possible for all wedge angles. This result can be understood on the basis that the normal tractions between the bodies or $\sigma_{\theta\theta}$ are not likely to vanish at the vertex of the wedge ($x = 0^+$) even for small wedge angles. In such case the shearing tractions $\sigma_{r\theta}$ have a jump discontinuity from $x = 0^+$ to $x = 0^-$ on basis of (2). It is known, however, that a jump discontinuity in shearing tractions leads to a logarithmic singularity in the tangential component of normal stress, or σ_{rr} in the present case.

Experimental Observations

Some photoelasticity experiments were done in order to see the difference in the isochromatic patterns caused by high versus low friction. The models were made of Hysol. The bottom body was a rectangular block. The top body with a sharp corner was pressed against the straight face of the bottom block. The coefficient of friction between the blocks was estimated to be near 1.0 under dry conditions. Lubrication was done with common machine grease. As no quantitative evaluation of the isochromatics is attempted, and the results are presented in a purely qualitative way, there is no need to describe the experimental arrangement in greater detail.

Figures 3 and 4 show relatively small areas of the overall isochromatic patterns. The difference between lubrication and dry conditions is quite noticeable. For smaller wedge angles there is a considerable change even in the global pattern. This becomes less pronounced for larger wedge angles, but the difference can be observed in the density and number of loops emanating from the singular point. In all cases the slip observed was in the direction of negative x, as defined previously, or the wedge slipping to the left with respect to the bottom block. The serrations that are seen in the isochromatics at the interface, especially under dry conditions, are due to nonuniform slip or local sticking.

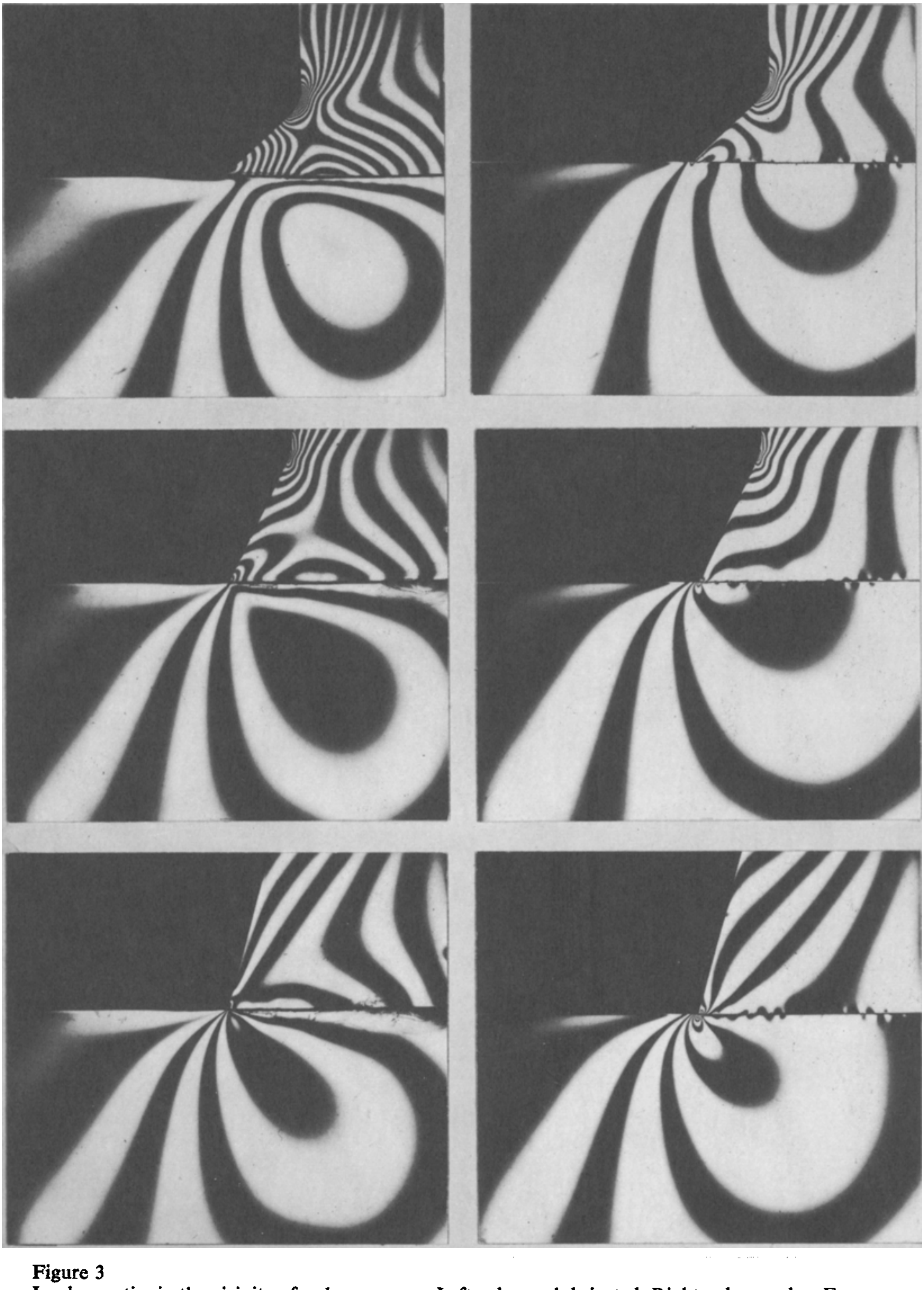

Figure 3
Isochromatics in the vicinity of a sharp corner. Left column: lubricated. Right column: dry. From top to bottom: $\gamma = 45°$, $60°$ and $75°$.

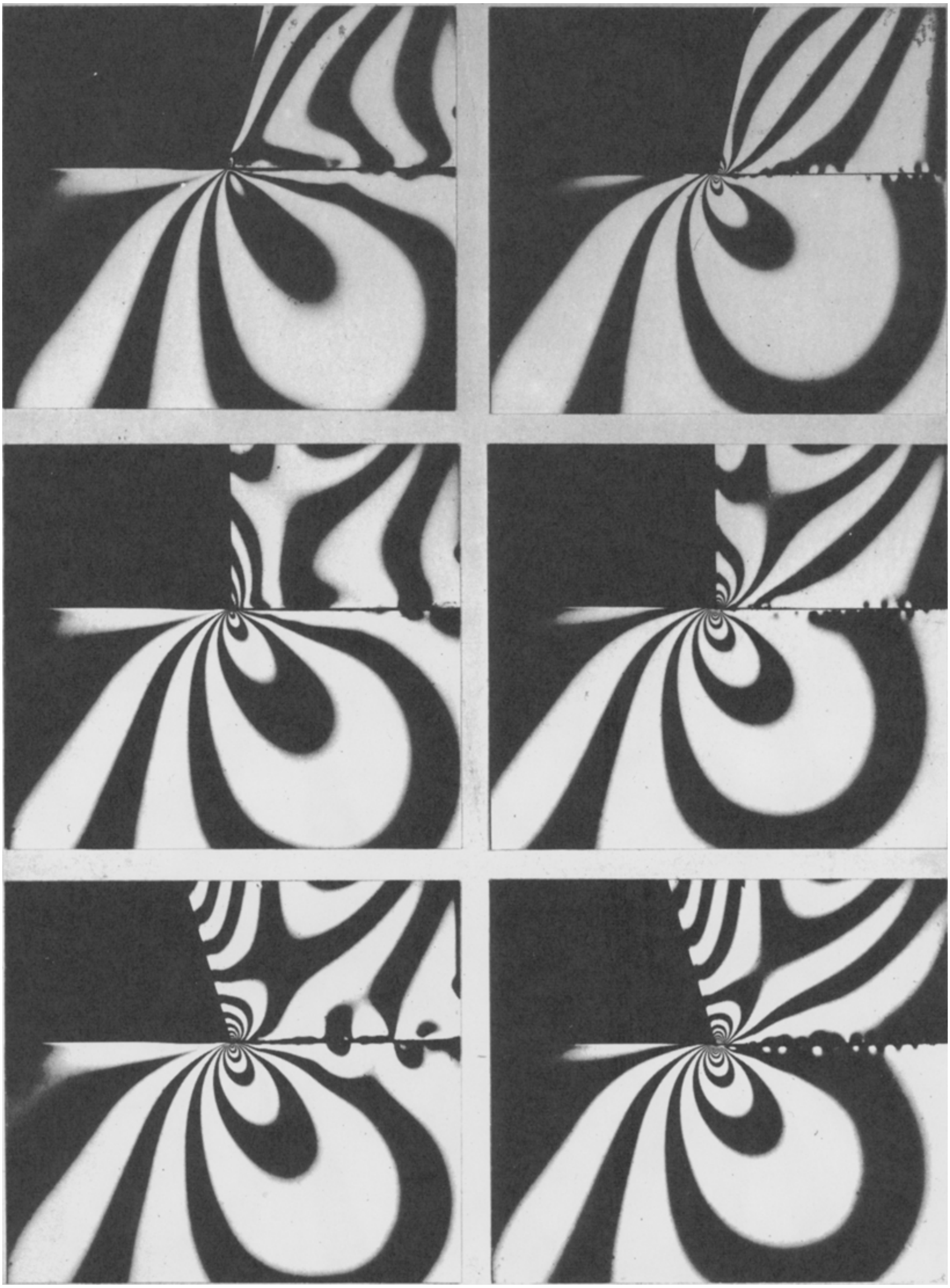

Figure 4
Continuation of Fig. 3. From top to bottom $\gamma = 77.5°$ (angle at which singularity starts to appear for identical materials and no friction), 90° and 105°.

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Abstract

The paper discusses the nature of the singularity that arises at a sharp edge in contact problems with friction. The theoretical treatment is based on the Mellin transform of the elastic fields. The results regarding the power singularities confirm the previous work of Gdoutos and Theocaris, but it is shown that logarithmic singularities are always present. Some experimental observations in photoelasticity are also presented.

Zusammenfassung

Die Art der Spannungssingularität, die an einer scharfen Ecke in Berührungsproblemen erscheint, ist für den Fall mit Reibung untersucht. Die theoretische Behandlung stützt sich auf die Mellin-Transformation der elastischen Felder. Die Ergebnisse bezüglich der Potenzsingularitäten bestätigen die früheren Resultate von Gdoutos und Theocaris. Es wird jedoch gezeigt, daß logarithmische Singularitäten stets anwesend sind. Auch einige Beobachtungen von photoelastischen Versuchen sind dargestellt.

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